Written Homework #04. Math 202 Due

Name: Solutions Section: ____________

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 3 \\ 1 & -4 & -1 \end{bmatrix}, \quad \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{\mathbf{c}} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

(a) Compute $T(\vec{\mathbf{u}})$.

1pt

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 3 \\ 1 & -4 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix} + (-1) \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$$

(b) Solve the equation $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$

solve $A\vec{x} = \vec{b}$

1pt

reduce $\left[\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ -1 & 2 & 3 & 2 \\ 1 & -4 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ r_2 + r_1 \\ r_3 - r_1 \end{array}$

$(0 \quad 3 \quad -3 \quad -3) \leftarrow (-3r_2)$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & -4 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} r_1 - 3r_2 \\ \\ r_3 - r_2 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x_1 \quad -5x_3 = -4 \\ x_2 - x_3 = -1 \\ x_3 \text{ free} \end{cases} \Leftrightarrow \begin{cases} x_1 = -4 + 5x_3 \\ x_2 = -1 + x_3 \\ x_3 \text{ free} \end{cases}$

(c) Is $\vec{\mathbf{c}}$ in the range of T? Justify your answer.

is $T(\vec{x}) = \vec{c}$ consistent?

1pt

is $\left[\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ -1 & 2 & 3 & 2 \\ 1 & -4 & -1 & 5 \end{array}\right]$ consistent?

$\sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 6 \end{array}\right]$ consistent? $\sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{array}\right]$

NOT consistent

$\boxed{\Rightarrow \vec{c} \text{ NOT in range of } T}$

Name: ______________________ Section: ______________________

2. (No Computation) Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

(a) What is the domain of T? $\mathbb{R}^3$

$A\vec{x}$ defined $\Leftrightarrow$ $\vec{x}$ in $\mathbb{R}^3$

(b) What is the co-domain of T? $\mathbb{R}^2$

$A\vec{x}$ has 2 rows $\Rightarrow A\vec{x}$ is in $\mathbb{R}^2$

(c) Describe the Range of T as the span of a set of vectors.

outputs are obtained by scaling & adding columns of A

$\text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\} = \text{Range}(T)$

3. (No Computation) How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^5$ into $\mathbb{R}^7$ by the rule $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$?

inputs $= n$ outputs $= m$

A must be a 7×5 matrix.

Written Homework #04. Math 202 Due

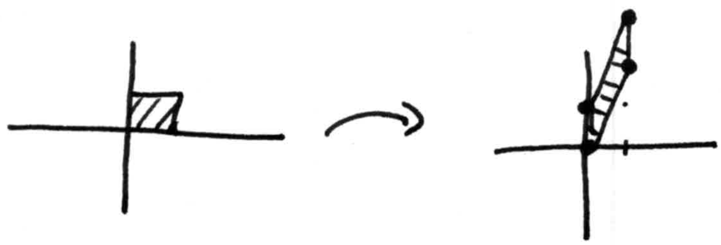
Name: ______________________ Section: ______________________

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

Using two axes (one for inputs and one for outputs), show how T transforms the vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Describe geometrically what the transformation T is doing (using words).

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 + x_2 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



vertical
Skew transformation

5. Find all the vectors $\vec{\mathbf{x}}$ that are mapped to $\vec{\mathbf{0}}$ by the transformation $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$
where $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$,

solve $T(\vec{x}) = \vec{0}$ $\iff$ Solve $A\vec{x} = \vec{0}$

$\iff$ reduce $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] r_3 - r_1$

$$\iff \begin{cases} x_1 + x_4 = 0 \\ x_2 + x_4 = 0 \\ x_3 - x_4 = 0 \end{cases}$$

x_4 free

$$\iff \begin{cases} x_1 = -x_4 \\ x_2 = -x_4 \\ x_3 = x_4 \\ x_4 \end{cases}.$$

So $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_4 \\ -x_4 \\ x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

Name: ______________________ Section: ______________________

9. What is the definition of a Linear Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$?

a transformation $T: \mathbb{R}^n \to \mathbb{R}^m$

so that

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$

and $T(c \cdot \vec{u}) = c \cdot T(\vec{u})$

10. Let $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{\mathbf{v}}_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, and $\vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\vec{\mathbf{x}}$ to $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2$. Find a matrix A so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ for every $\vec{\mathbf{x}}$.

$$T(\vec{x}) = x_1\vec{v}_1 + x_2\vec{v}_2$$

$$= x_1\begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ -5 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ -5 & -2 \end{bmatrix}$$

Name: ______________________ Section: ______________________

8. Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that sends $\vec{\mathbf{e}}_1$ to $\vec{\mathbf{e}}_1 - 3\vec{\mathbf{e}}_2$ and leaves $\vec{\mathbf{e}}_2$ unchanged.

$$T(\vec{e}_1) = \vec{e}_1 - 3\vec{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$T(\vec{e}_2) = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

Name: ____________ Section: ____________

11. Prove that $T(\vec{\mathbf{x}}) = 3\vec{\mathbf{x}}$ is a linear transformation.

0.5 pt

Recall T is linear $\iff$ $T(c\vec{u} + d\vec{v}) = c \cdot T(\vec{u}) + d \cdot T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$

$P \leftrightarrow Q$

Proof by computation

0.5 pt

$$T(c\vec{u} + d\vec{v}) = 3 \cdot (c\vec{u} + d\vec{v})$$
$$= c \cdot 3\vec{u} + d \cdot 3\vec{v}$$
$$= c \cdot T(\vec{u}) + d \cdot T(\vec{v}) \checkmark$$

$$\frac{Q}{P}$$

12. Prove that $T(\vec{\mathbf{x}}) = 3\vec{\mathbf{x}} + 1$ is not linear transformation.

Recall: T is linear $\iff$ $T(c\vec{u} + d\vec{v}) = c \cdot T(\vec{u}) + d \cdot T(\vec{v})$

$P \leftrightarrow Q$

0.5 pt

proof by computation

$$T(c\vec{u} + d\vec{v}) = 3(c\vec{u} + d\vec{v}) + 1$$
$$= c \cdot 3\vec{u} + d \cdot 3\vec{v} + 1$$

0.5 pt

← cannot "find" $c \cdot T(\vec{u})$ or $d \cdot T(\vec{v})$ here

$$\neq c \cdot T(u) + d \cdot T(v)$$

So T is not linear.

$$\frac{\neg Q}{\neg P}$$

Name: ______________________ Section: ______________________

3 assumptions

13. Suppose that $\boxed{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n \text{ span } \mathbb{R}^n}$ and that $\boxed{T \text{ is a linear transformation}}$ with $\boxed{T(\vec{\mathbf{v}}_1) = \vec{\mathbf{0}}, \ldots, T(\vec{\mathbf{v}}_n) = \vec{\mathbf{0}}.}$ Prove that $T(\vec{\mathbf{b}}) = \vec{\mathbf{0}}$ for *every* $\vec{\mathbf{b}} \in \mathbb{R}^n$. ← cannot use this in your proof

proof by computation.

introduce suitable Notation

let $\vec{b}$ be any vector in $\mathbb{R}^n$

by definition of $\mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_n\} = \mathbb{R}^n$

$$\vec{b} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

Compute

$$T(\vec{b}) = T(c_1\vec{v}_1 + \cdots + c_n \cdot \vec{v}_n)$$

by linearity

$$= c_1 \cdot T(\vec{v}_1) + \cdots + c_n \cdot T(v_n)$$

because $T(v_1) = \cdots = T(v_n) = \vec{0}$

$$= c_1 \cdot \vec{0} + \cdots + c_n \cdot \vec{0}$$

$$T(\vec{b}) = \vec{0}$$

← as desired.

1 Bonus point if complete & correct

Name: ______________________ Section: ______________________

1. Let transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with
$T(\vec{\mathbf{e}}_1) = \begin{bmatrix}1\\3\end{bmatrix}$, $T(\vec{\mathbf{e}}_2) = \begin{bmatrix}-1\\3\end{bmatrix}$ and $T(\vec{\mathbf{e}}_3) = \begin{bmatrix}2\\4\end{bmatrix}$.

(a) Find the standard matrix A of T.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

(b) Determine if the transformation T is one-to-one.

T is one-to-one $\Leftrightarrow$ columns of A are indep

$\Leftrightarrow$ $A\vec{x} = \vec{0}$ has unique solution

reduce $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 3 & 3 & 4 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 6 & -2 & 0 \end{array}\right] r_2 - 3r_1$

$\underbrace{\hphantom{1 \quad -1 \quad 2}}_{\text{reduced matrix } A}$ $\nwarrow$ ∞-many solutions $\Rightarrow$ nontrivial sln

$\Rightarrow$ cols <u>NOT</u> indep.

$\Rightarrow$ T is <u>NOT</u> one-to-one

(c) Determine if the transformation T is onto.

T is onto $\Leftrightarrow$ columns of A span $\mathbb{R}^2$

<u>NOTICE</u> pivot in every row of reduced echelon form of A

$\Rightarrow$ columns of A span $\mathbb{R}^2$

$\Rightarrow$ T is onto

Name: ____________________ Section: ____________________

- Let transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with

$$T(\vec{\mathbf{e}}_1) = \begin{bmatrix}1\\0\\0\end{bmatrix},\ T(\vec{\mathbf{e}}_2) = \begin{bmatrix}-1\\3\\0\end{bmatrix} \text{ and } T(\vec{\mathbf{e}}_3) = \begin{bmatrix}0\\4\\1\end{bmatrix}.$$

(a) Find the standard matrix A of T.

1 pt

$$A = \begin{bmatrix}1 & -1 & 0\\0 & 3 & 4\\0 & 0 & 1\end{bmatrix}$$

(b) Determine if the transformation T is one-to-one.

$P \Leftrightarrow Q$ $\qquad$ T is one-to-one $\Leftrightarrow$ col's of A are independent

$\Leftrightarrow A\vec{x} = \vec{0}$ has unique solution

1 pt

$$\left[\begin{array}{ccc|c}1 & -1 & 0 & 0\\0 & 3 & 4 & 0\\0 & 0 & 1 & 0\end{array}\right]$$

already in echelon form

note: pivot in each column of A

$Q \Rightarrow$ unique solution

$\Rightarrow$ no nontrivial solutions

$\Rightarrow$ col's of A are indep

$P \Rightarrow T$ is one-to-one

(c) Determine if the transformation T is onto.

T is onto $\Leftrightarrow$ columns of A span $\mathbb{R}^3$

Notice: in reduced form of A, pivot in every row

1 pt

$\Rightarrow$ columns of A span $\mathbb{R}^3$

$\Rightarrow T$ is onto.

Name: ____________ Section: ____________

14. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with standard form matrix A. A is 3×2

Prove that T is not onto. (Cite all relevant definitions and theorems by number).

proof by rephrasing

T is onto

$P \Leftrightarrow Q$ $\Leftrightarrow$ Theorem 12a

columns of A span $\mathbb{R}^3$

$Q \Leftrightarrow R$ $\Leftrightarrow$ Theorem 4

pivot in each row of A

But $\neg R$ $\boxed{\text{you cannot have 3 pivots in only 2 columns}}$

1 pt

$\neg P$ So T is not onto

15. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with standard form matrix A.

Given this info do you know if ~~Can you prove that~~ T is onto? Justify your answer.

Hint: answer is "No, you cannot know if T is onto or not onto"

why?

a 2x3 A might have a pivot in each row (e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$)

but also

A might not have a pivot in each row (e.g. $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}$).

Name: ____________________ Section: ____________________

Definitions

1. Define Span$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

$$= \left\{ \vec{b} \quad \text{s.t.} \quad \vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \text{ for } c_1, c_2, c_3 \text{ in } \mathbb{R} \right\}$$

2. Define linear **Independence** of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

the set is independent
$\Longleftrightarrow$
$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ has ONLY the trivial solution

3. Define "T is a linear transformation"

$\Longleftrightarrow$
it satisfies
$$\left.\begin{aligned} T(\vec{u}+\vec{v}) &= T(\vec{u}) + T(\vec{v}) \\ T(c\cdot\vec{u}) &= c\cdot T(\vec{v}) \end{aligned}\right\} \text{ for all } \vec{u}, \vec{v} \text{ \& } c$$

4. Define "T is one-to-one"

$\Longleftrightarrow$
$T(\vec{x}) = \vec{b}$ has at most one solution
for each $\vec{b}$

5. Define "T is onto"

$\Longleftrightarrow$
$T(\vec{x}) = \vec{b}$ has at least one solution
for each $\vec{b}$

Name: ______________________ Section: ______________________

Theorems

Theorem 2 The reduced echelon form of a linear system has three possible cases

1. The system has 0 solutions if it contains $[0 \cdots 0 \mid \blacksquare]$
2. The system has 1 solutions if it has pivot in each variable column
3. The system has ∞-many solutions if it has variable column w/o pivot

Theorem 4: The columns of an $m \times n$ matrix A span $\mathbb{R}^m$

if and only if there is a pivot in each row

Theorem 5 If A is an $m \times n$ matrix, $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, *Then*

- $A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) =$ $A\vec{u} + A\vec{v}$ i.e. $T(\vec{x}) = A\vec{x}$ is linear.
- $A(c \cdot \vec{\mathbf{u}}) =$ $c \cdot A\vec{u}$

Properties of Linear Transformations

- If T is linear, then $T(\vec{\mathbf{0}}) =$ $\vec{0}$
- T is linear $\iff$ $T(c \cdot \vec{\mathbf{u}} + d \cdot \vec{\mathbf{v}}) =$ $c \cdot T(\vec{u}) + d \cdot T(\vec{v})$

Theorem 10 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear.

Then there is a unique $m \times n$ matrix A s.t. $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

In Fact, $A =$ $\left[T(\vec{e}_1) \cdots T(\vec{e}_n) \right]$ where $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Theorem 12 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear with standard matrix A. T

(a) T is onto $\iff$ columns of A span $\mathbb{R}^m$

(b) T is one-to-one $\iff$ columns of A are linearly independent